

Error estimates in balanced norms of finite element methods on Shishkin meshes for reaction-diffusion problems

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Abstract

Error estimates of finite element methods for reaction-diffusion problems are often realized in the related energy norm. In the singularly perturbed case, however, this norm is not adequate. A different scaling of the H^1 seminorm leads to a balanced norm which reflects the layer behavior correctly. We discuss also anisotropic problems, semilinear equations, supercloseness and a combination technique.

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1 Introduction

We shall examine the finite element method for the numerical solution of the singularly perturbed linear elliptic boundary value problem

$$(1.1a) \quad Lu \equiv -\varepsilon \Delta u + cu = f \quad \text{in } \Omega = (0, 1) \times (0, 1)$$

$$(1.1b) \quad u = 0 \quad \text{on } \partial\Omega,$$

where $0 < \varepsilon \ll 1$ is a small positive parameter, $c > 0$ is (for simplicity) a positive constant and f is sufficiently smooth.

It is well-known that the problem has a unique solution $u \in V = H_0^1(\Omega)$ which satisfies the stability estimate in the related energy norm

$$(1.2) \quad \|u\|_\varepsilon := \varepsilon^{1/2} |u|_1 + \|u\|_0 \preceq \|f\|_0.$$

Here we used the following notation: if $A \preceq B$, there exists a (generic) constant C independent of ε (and later also of the mesh used) such that $A \leq C B$. Moreover for $D \subset \Omega$ we denote by $\|\cdot\|_{0,D}$, $\|\cdot\|_{\infty,D}$ and $|\cdot|_{1,D}$ the standard norms in $L_2(D)$, $L_\infty(D)$ and the standard seminorm in $H^1(D)$, respectively. We shall omit the notation of the domain in the case $D = \Omega$. Similarly, we want to use the notation $(\cdot, \cdot)_D$ for the inner product in $L_2(D)$ and abbreviate $(\cdot, \cdot)_\Omega$ to (\cdot, \cdot) .

Moreover, the error of a finite element approximation $u^N \in V^N \subset V$ satisfies

$$(1.3) \quad \|u - u^N\|_\varepsilon \preceq \min_{v^N \in V^N} \|u - v^N\|_\varepsilon.$$

When linear or bilinear elements are used on a Shishkin mesh (see Section 2), one can prove under certain additional assumptions concerning f for the interpolation error of the Lagrange interpolant $u^I \in V^N$ on Shishkin meshes

$$(1.4) \quad \|u - u^I\|_\varepsilon \preceq (\varepsilon^{1/4} N^{-1} \ln N + N^{-2})$$

(see [5] or [18]). It follows that the error $u - u^N$ also satisfies such an estimate.

However, the typical boundary layer function $\exp(-x/\varepsilon^{1/2})$ measured in the norm $\|\cdot\|_\varepsilon$ is of order $\mathcal{O}(\varepsilon^{1/4})$. Consequently, error estimates in this norm are less valuable as for convection diffusion equations where the layers are of the structure $\exp(-x/\varepsilon)$. Wherefore we ask the fundamental question:

Is it possible to prove error estimates in the balanced norm

$$(1.5) \quad \|v\|_b := \varepsilon^{1/4} |v|_1 + \|v\|_0 \quad ?$$

In Section 2 we will repeat an basic idea to prove error estimates in a balanced norm and extend the approach to semilinear problems and anisotropic equations. Supercloseness and a combination technique are discussed in Section 3. Finally we present a direct mixed method in Section 4.

2 The basic error estimate in a balanced norm and some extensions

The mesh Ω^N used is the tensor product of two one-dimensional piecewise uniform Shishkin meshes. I.e., $\Omega^N = \Omega_x \times \Omega_y$, where Ω_x (analogously Ω_y)

splits $[0, 1]$ into the subintervals $[0, \lambda_x]$, $[\lambda_x, 1 - \lambda_x]$ and $[1 - \lambda_x, 1]$. The mesh distributes $N/4$ points equidistantly within each of the subintervals $[0, \lambda_x]$, $[1 - \lambda_x, 1]$ and the remaining points within the third subinterval. For simplicity, assume

$$\lambda = \lambda_x = \lambda_y = \min\{1/4, \lambda_0 \sqrt{\varepsilon/c^*} \ln N\} \quad \text{with } \lambda_0 = 2 \text{ and } c^* < c.$$

We use for the step sizes

$$h := \frac{4\lambda}{N} \quad \text{and} \quad H := \frac{2(1 - 2\lambda)}{N}.$$

Let $V^N \subset H_0^1(\Omega)$ be the space of bilinear finite elements on Ω^N or the space of linear elements over a triangulation obtained from Ω^N by drawing diagonals.

A standard formulation of problem (3.1) reads: find $u \in V$, such that

$$(2.1) \quad \varepsilon(\nabla u, \nabla v) + c(u, v) = (f, v) \quad \forall v \in V.$$

By replacing V in (2.8) with V^N one obtains a standard discretization that yields the FEM-solution u^N .

As we mentioned already in the Introduction, certain assumptions on f allow a decomposition of u into smooth components and layer terms such that the following estimates for the interpolation error of the Lagrange interpolant hold true (see [5] or [18]):

$$(2.2) \quad \|u - u^I\|_0 \preceq N^{-2}, \quad \varepsilon^{1/4}|u - u^I|_1 \preceq N^{-1} \ln N$$

and

$$(2.3) \quad \|u - u^I\|_{\infty, \Omega_0} \preceq N^{-2}, \quad \|u - u^I\|_{\infty, \Omega \setminus \Omega_0} \preceq (N^{-1} \ln N)^2,$$

here $\Omega_0 = (\lambda_x, 1 - \lambda_x) \times (\lambda_y, 1 - \lambda_y)$. Let us also introduce $\Omega_f := \Omega \setminus \Omega_0$.

Instead of the Lagrange interpolant we use in our error analysis the L_2 projection $\pi u \in V^N$ from u . Based on

$$u - u^N = u - \pi u + \pi u - u^N$$

we estimate $\xi := \pi u - u^N$:

$$\|\xi\|_\varepsilon^2 \preceq \varepsilon \|\nabla \xi\|_1^2 + c \|\xi\|_0^2 = \varepsilon(\nabla(\pi u - u), \nabla \xi) + c(\pi u - u, \xi).$$

Because $(\pi u - u, \xi) = 0$, it follows

$$(2.4) \quad |\pi u - u^N|_1 \preceq |u - \pi u|_1.$$

If we now could prove a similar estimate as (4.2) for the error of the L_2 projection, we obtain an estimate in the balanced norm because we have already an estimate for $\|u - u_N\|_0$ in (3.2).

Lemma 1. *Assuming the validity of (4.2) and (2.3), the error of the L_2 projection on the Shishkin mesh satisfies*

$$(2.5) \quad \|u - \pi u\|_\infty \preceq \|u - u^I\|_\infty, \quad \varepsilon^{1/4} |u - \pi u|_1 \preceq N^{-1} (\ln N)^{3/2}.$$

The proof uses the L_∞ -stability of the L_2 projection on our mesh [17]. Inverse inequalities are used to move from estimates in W_∞^1 to L_∞ , for details see [19].

From (4.4) and Lemma 1 we get

Theorem 1:. *Assuming (4.2) and (2.3), the error of the Galerkin finite element method with linear or bilinear elements on a Shishkin mesh satisfies*

$$(2.6) \quad \|u - u^N\|_b \preceq N^{-1} (\ln N)^{3/2} + N^{-2}.$$

Remark that for Q_k elements with $k > 1$ one can get an analogous result

$$\|u - u^N\|_b \preceq N^{-k} (\ln N)^{k+1/2} + N^{-(k+1)}$$

because on tensor product meshes the L_2 projection is as well L_∞ stable (see [4] for the one-dimensional result on arbitrary meshes, on tensor product meshes the statement follows immediately).

It is easy to modify the basic idea to the singularly perturbed semilinear elliptic boundary value problem

$$(2.7a) \quad Lu \equiv -\varepsilon \Delta u + g(\cdot, u) = 0 \quad \text{in } \Omega = (0, 1) \times (0, 1)$$

$$(2.7b) \quad u = 0 \quad \text{on } \partial\Omega.$$

We assume that g is sufficiently smooth and $\partial_2 g \geq \mu > 0$. Then, the so called reduced problem and our given problem have a unique solution.

If $\partial\Omega$ is smooth, the solution is characterized by the typical boundary layer for linear reaction-diffusion problems, see [8] for the semilinear case.

If corners exist, additionally corner layers arise, see [9] for semilinear problems in a polygonal domain. For the analysis of finite element methods on layer-adapted meshes we need a solution decomposition (see Remark 1.27 in Chapter 3 of [18]), in the semilinear case sufficient conditions for the existence of such a decomposition are not known. Therefore we just assume the existence of a solution decomposition.

A standard weak formulation of our semilinear problem reads: find $u \in V$, such that

$$(2.8) \quad \varepsilon(\nabla u, \nabla v) + (g(\cdot, u), v) = 0 \quad \forall v \in V.$$

By replacing V in (2.8) with V^N one obtains a standard discretization that yields the FEM-solution u^N .

If $\pi u \in V^N$ is some projection of u , we decompose the error into

$$u - u^N = u - \pi u + \pi u - u^N$$

and (assuming we can control the projection error) start the error analysis from the following relation for $\xi := \pi u - u^N$:

$$\varepsilon|\nabla \xi|_1^2 + \mu \|\xi\|_0^2 \leq \varepsilon(\nabla \xi, \nabla \xi) + (g(\cdot, \pi u) - g(\cdot, u^N), \xi) = \varepsilon(\nabla(\pi u - u), \nabla \xi) + (g(\cdot, \pi u) - g(\cdot, u), \xi).$$

If we choose πu to be the standard interpolant of u , the usual error estimate in the energy norm

$$(2.9) \quad \|u\|_\varepsilon := \varepsilon^{1/2}|u|_1 + \|u\|_0$$

follows:

$$(2.10) \quad \|u - u^N\|_\varepsilon \preceq (\varepsilon^{1/4} N^{-1} \ln N + N^{-2})$$

But we want again to prove an error estimate in the balanced norm

$$(2.11) \quad \|v\|_b := \varepsilon^{1/4}|v|_1 + \|v\|_0 \quad .$$

Following the basic idea from [19], we define πu by

$$(2.12) \quad (g(\cdot, \pi u), v) = (g(\cdot, u), v) \quad \text{for all } v \in V^N.$$

Our assumption $\partial_2 g \geq \mu > 0$ tells us immediately that πu is well defined, moreover

$$(2.13) \quad \|u - \pi u\|_0 \preceq \inf_{v \in V^N} \|u - v\|_0.$$

It follows from the definition of our projection

$$(2.14) \quad |\pi u - u^N|_1 \preceq |u - \pi u|_1.$$

For the standard interpolant u^I of u we have

$$\varepsilon^{1/4} |u - u^I|_1 \preceq N^{-1} \ln N.$$

If we now could prove a similar estimate for our projection error, we would obtain an estimate in the balanced norm because we have already an estimate for $\|u - u_N\|_0$ in (2.10).

Lemma 2. *The projection defined by (2.12) is L_∞ stable.*

Proof. The proof is based on Taylors formula

$$F(w) - F(v) = \left(\int_0^1 DF(v + s(w - v)) ds \right) (w - v).$$

Introducing the linear operator

$$\Delta F(v, w) := \int_0^1 DF(v + s(w - v)) ds$$

it is obvious that

$$\|w - v\| \leq \|(\Delta F(v, w))^{-1}\| \|F(w) - F(v)\|.$$

Therefore, the L_∞ stability of the L_2 projection on our mesh [17] implies the L_∞ stability of our generalized projection as well.

Lemma 3. *The projection error of (2.12) on the Shishkin mesh satisfies*

$$(2.15) \quad \|u - \pi u\|_\infty \preceq \|u - u^I\|_\infty, \quad \varepsilon^{1/4} |u - \pi u|_1 \preceq N^{-1} (\ln N)^{3/2}.$$

The proof works analogously as in the linear case. And, consequently, we get the same error estimate as in Theorem 1 also in the semilinear case.

Next we consider the anisotropic problem

$$(2.16a) \quad -\varepsilon u_{xx} + u_{yy} + cu = f \quad \text{in } \Omega = (0, 1) \times (0, 1)$$

$$(2.16b) \quad u = 0 \quad \text{on } \partial\Omega.$$

Now we have only boundary layers at $x = 0$ and $x = 1$, the layers are of elliptic type. But the layer terms satisfy the same estimates as in the reaction-diffusion regime [11]. Therefore, the estimates (2.2) and (2.3) for the interpolation error on the related Shishkin mesh remain valid, of course, now $\Omega_0 = (\lambda_x, 1 - \lambda_x) \times (0, 1)$. Therefore, defining the energy norm by

$$\|v\|_{\varepsilon,a} := \varepsilon^{1/2} \|u_x\|_0 + \|u_y\|_0 + \|u\|_0$$

it follows for bilinear elements

$$\|u - u^N\|_{\varepsilon,a} \preceq (\varepsilon^{1/4} N^{-1} \ln N + N^{-2}).$$

If we want to estimate the error in the balanced norm

$$\|v\|_{b,a} := \varepsilon^{1/4} \|u_x\|_0 + \|u_y\|_0 + \|u\|_0,$$

we start for $\xi := \pi u - u^N$ from

$$\varepsilon \|\xi_x\|_0^2 \leq \varepsilon ((\pi u - u)_x, \xi_x) + ((\pi u - u)_y, \xi_y) + c (\pi u - u, \xi).$$

Now we define in the anisotropic case the projection onto the finite element space by

$$((\pi u - u)_y, \xi_y) + c (\pi u - u, \xi) = 0 \quad \forall \xi \in V^N.$$

Consequently it remains to estimate for that projection $\|(\pi u - u)_x\|_0$. But the projection satisfies

$$\pi v = \pi^y(\pi^x v),$$

where π^x is the one-dimensional L_2 projection and π^y the one-dimensional Ritz projection (with respect to a non-singularly perturbed operator on a standard mesh), compare [7]. Consequently, the projection is L_∞ stable and we can repeat our basic idea to prove estimates in the balanced norm.

Remark that in [16] the authors use a different technique to derive estimates in the H^1 seminorm for the generalized L_2 projection on a layer-adapted mesh.

3 Supercloseness and a combination technique

We come back to the linear reaction-diffusion problem

$$(3.1a) \quad Lu \equiv -\varepsilon \Delta u + cu = f \quad \text{in } \Omega = (0, 1) \times (0, 1)$$

$$(3.1b) \quad u = 0 \quad \text{on } \partial\Omega.$$

For bilinear elements on the corresponding Shishkin mesh it is well known that we have the supercloseness property (assuming $\lambda_0 \geq 2.5$)

$$(3.2) \quad \|u^N - u^I\|_\varepsilon \preceq (\varepsilon^{1/2}(N^{-1} \ln N)^2 + N^{-2}).$$

Now we ask: does there exist some projection onto the finite element space such that a supercloseness property holds with respect to the balanced norm?

With $v_N := u^N - \Pi u$ we start from

$$\varepsilon |v_N|_1^2 + c \|v_N\|_0^2 \preceq \varepsilon (\nabla(u - \Pi u), \nabla v_N) + c(u - \Pi u, v_N).$$

Next we use the decomposition $u = S + E$, decompose also $\Pi u = \Pi S + \Pi E$ and use different projections into our bilinear finite element space for S and E . We choose:

- $\Pi S \in V^N$ satisfies

$$(\Pi S, v) = (S, v) \quad \forall v \in V_0^N$$

with given values in the grid points on the boundary.

- ΠE is zero in Ω_0 and the standard bilinear interpolation operator in the fine subdomain with exception of one strip of the width of the fine stepsize in the transition region (and, of course, bilinear in that strip and globally continuous)

With this choice we obtain

$$\varepsilon |v_N|_1^2 + c \|v_N\|_0^2 \preceq \varepsilon (\nabla(u - \Pi u), \nabla v_N) + c(E - \Pi E, v_N)_{\Omega_f}.$$

In the second term we hope to get some extra power of ε , in the first term we want to apply superconvergence techniques for the estimation of the expression $(\nabla(E - \Pi E), \nabla v_N)$. First let us remark that ΠE satisfies the same estimates as the bilinear interpolant E^I on Ω_f :

$$\|E - \Pi E\|_{0, \Omega_f} \preceq \varepsilon^{1/4}(N^{-1} \ln N)^2$$

and (based on Lin identities)

$$\varepsilon |(\nabla(E - \Pi E), \nabla v_N)| \preceq N^{-2} \varepsilon^{3/4} |v_N|_1.$$

It is only a technical question to prove that for our modified interpolant using the fact that E is on that strip is as small as we want and that the measure of the strip is small as well.

Consequently we get

$$|v_N|_1^2 \preceq |S - \Pi S|_1^2 + \varepsilon^{-1/2} (N^{-1} \ln N)^4.$$

For the L_2 projection of S we have $\|S - \Pi S\|_\infty \preceq N^{-2}$ and $\|S - \Pi S\|_{\infty, \Omega_f} \preceq (\varepsilon^{1/2} N^{-1} \ln N)^2$. It follows

$$|S - \Pi S|_{1, \Omega_0} \preceq N^{-1}, \quad |S - \Pi S|_{1, \Omega_f} \preceq \varepsilon^{1/2} N^{-1} \ln N.$$

Summarizing we get the supercloseness result

$$\varepsilon^{1/4} |u^N - \Pi u|_1 \preceq \varepsilon^{1/4} N^{-1} + (N^{-1} \ln N)^2.$$

It is no problem to estimate the L_2 error.

Next we present an application of the supercloseness result to the combination technique. We analyse the version of the combination technique presented in [6], for a different version see [14]. Remark that in [15] the authors observe numerically a nice behaviour of a combination technique in the balanced norm.

Writing N for the maximum number of mesh intervals in each coordinate direction, our combination technique simply adds or subtracts solutions that have been computed by the Galerkin FEM on $N \times \sqrt{N}$, $\sqrt{N} \times N$ and $\sqrt{N} \times \sqrt{N}$ meshes. We obtain the same accuracy as on a $N \times N$ mesh with less degrees of freedom. In the following we use the notation of [6].

In the combination technique for bilinear elements we compute a two-scale finite element approximation $u_{\hat{N}, \hat{N}}^N$ by

$$u_{\hat{N}, \hat{N}}^N := u_{N, \hat{N}}^N + u_{\hat{N}, N}^N - u_{\hat{N}, \hat{N}}^N.$$

Later we will choose $\hat{N} = \sqrt{N}$. We proved (in our new notation)

$$(3.3) \quad \|u - u_{NN}\|_b \preceq N^{-1} (\ln N)^{3/2} + N^{-2}.$$

The question is whether or not $u_{\hat{N},\hat{N}}^N$ satisfies a similar estimate (in the case $\hat{N} = \sqrt{N}$).

Analogously to $u_{\hat{N},\hat{N}}^N$ we define $I_{\hat{N},\hat{N}}^N E$ and $\Pi_{\hat{N},\hat{N}}^N S$. Then we can as follows decompose the error to estimate:

$$u_{\hat{N},\hat{N}}^N - u_{NN} = T_{cl,1}(S) + (\Pi_{\hat{N},\hat{N}}^N S - \Pi_{N,N} S) + T_{cl,s}(E) + (I_{\hat{N},\hat{N}}^N E - I_{N,N} E)$$

Thus we have two terms representing the error for two-scale projection operators (related to L_2 projection and interpolation, respectively) and two terms which can be estimated based on our supercloseness result:

$$T_{cl,1}(S) := (S_{N,\hat{N}} - \Pi_{N,\hat{N}} S) + (S_{\hat{N},N} - \Pi_{\hat{N},N} S) - (S_{\hat{N},\hat{N}} - \Pi_{\hat{N},\hat{N}} S),$$

analogously

$$T_{cl,2}(E) := (E_{N,\hat{N}} - I_{N,\hat{N}} E) + (E_{\hat{N},N} - I_{\hat{N},N} E) - (E_{\hat{N},\hat{N}} - I_{\hat{N},\hat{N}} E),$$

For the two-scale interpolation error $(I_{\hat{N},\hat{N}}^N E - I_{N,N} E)$ the results of [6] remain valid (Lemma 2.3 and 2.4, modified for the reaction-diffusion problem). For the two-scale projection error an estimate in L_2 and L_∞ is easy. The estimate in the seminorm $|\cdot|$ as in Section 2 follows from an inverse inequality, applied separately in Ω_0 and Ω_f . Finally we get for $\hat{N} = \sqrt{N}$ the estimate

$$(3.4) \quad \|u_{\hat{N},\hat{N}}^N - u_{NN}\|_b \preceq \varepsilon^{1/4} N^{-1/2} + N^{-1} \ln N.$$

That means so far we can only proof the desired estimate for the combination technique if $\varepsilon \preceq N^{-2}$.

4 A direct mixed method

The first balanced error estimate was presented by Lin and Stynes [13] using a first order system least squares (FOSLS) mixed method. For the variables (u, \bar{q}) with $-\bar{q} = \nabla u$ and its discretizations on a Shishkin mesh they proved

$$(4.1) \quad \varepsilon^{1/4} |\bar{q} - \bar{q}^N|_1 + \|u - u^N\|_0 \preceq N^{-1} \ln N$$

(see also [1] for a modified version of the method)

We shall proof that the estimate (4.1) is also valid for a direct mixed method (instead the more complicated least-squares approach from [13]).

Remark that Li and Wheeler [12] analyzed the method in the energy norm on so called A-meshes, simpler to analyze than S-meshes.

Introducing $\bar{q} = -\nabla u$, a weak formulation of (3.1) reads:

Find $(u, \bar{q}) \in V \times W$ such that

$$(4.2a) \quad \varepsilon(\operatorname{div} \bar{q}, w) + c(u, w) = (f, w) \quad \text{for all } w \in W,$$

$$(4.2b) \quad \varepsilon(\bar{q}, \bar{v}) - \varepsilon(\operatorname{div} \bar{v}, u) = 0 \quad \text{for all } \bar{v} \in V,$$

with $V = H(\operatorname{div}, \Omega)$, $W = L^2(\Omega)$.

For the discretization on a standard rectangular Shishkin mesh (see [13], page 2735) we use $(u^N, \bar{q}^N) \in V^N \times W^N$. Here W^N is the space of piecewise constants on our rectangular mesh and V^N the lowest order Raviart-Thomas space RT_0 . That means, on each mesh rectangle elements of RT_0 are vectors of the form

$$(\operatorname{span}(1, x), \operatorname{span}(1, y))^T.$$

Our discrete problem reads: Find $(u^N, \bar{q}^N) \in V^N \times W^N$ such that

$$(4.3a) \quad \varepsilon(\operatorname{div} \bar{q}^N, w) + c(u^N, w) = (f, w) \quad \text{for all } w \in W^N,$$

$$(4.3b) \quad \varepsilon(\bar{q}^N, \bar{v}) - \varepsilon(\operatorname{div} \bar{v}, u^N) = 0 \quad \text{for all } \bar{v} \in V^N.$$

Setting $w := u^N$, $\bar{v} := \bar{q}^N$ results in the stability estimate

$$(4.4) \quad \varepsilon \|\bar{q}^N\|_0^2 + \frac{c}{2} \|u^N\|_0^2 \preceq \|f\|_0^2.$$

The unique solvability of the discrete problem follows (if $f \equiv 0$).

For the error estimation we introduce projections $\Pi : V \mapsto V^N$ and $P : W \mapsto W^N$. As usual, instead of $u - u^N$ and $\bar{q} - \bar{q}^N$ we estimate $Pu - u^N$ and $\Pi\bar{q} - \bar{q}^N$, assuming that we can estimate the projection errors. Subtraction of the continuous and the discrete problem results in

$$(4.5a) \quad \varepsilon(\nabla \cdot (\Pi\bar{q} - \bar{q}^N), w) + c(Pu - u^N, w) = \varepsilon(\nabla \cdot (\Pi\bar{q} - \bar{q}), w) + c(Pu - u, w),$$

$$(4.5b) \quad \varepsilon(\Pi\bar{q} - \bar{q}^N, \bar{v}) - \varepsilon(\nabla \cdot \bar{v}, Pu - u^N) = \varepsilon(\Pi\bar{q} - \bar{q}, \bar{v}) - \varepsilon(\nabla \cdot \bar{v}, Pu - u).$$

Setting $\bar{v} := \Pi\bar{q} - \bar{q}^N = \bar{\mu}$ and $w := Pu - u^N = \tau$ we obtain the error equation

$$(4.6) \quad \varepsilon(\bar{\mu}, \bar{\mu}) + c(\tau, \tau) = \varepsilon(\nabla \cdot (\Pi\bar{q} - \bar{q}), \tau) + c(Pu - u, \tau) + \varepsilon(\Pi\bar{q} - \bar{q}, \bar{\mu}) - \varepsilon(\nabla \cdot \bar{\mu}, Pu - u).$$

From the error equation it is easy to derive a first order uniform convergence result in the energy norm (one could also think about supercloseness similar as in [12]). But we want to investigate, whether or not an estimate of the type (4.1) is possible.

If P denotes the L_2 projection, we have

$$(Pu - u, \tau) = 0 \quad \text{and} \quad (\nabla \cdot \bar{\mu}, Pu - u) = 0,$$

because $\nabla \cdot \bar{\mu}$ is piecewise constant for $\bar{\mu} \in V^N$. Therefore, from the right hand side of the error equation two terms disappear and it follows

$$(4.7) \quad \|\bar{\mu}\|_0^2 \preceq \varepsilon \|\nabla \cdot (\Pi(\nabla u) - \nabla u)\|_0^2 + \|\Pi(\nabla u) - \nabla u\|_0^2.$$

Now let us denote by Π^* the standard local projection operator into the Raviart-Thomas space V^N . This operator satisfies

$$(4.8) \quad (\nabla \cdot (\bar{v} - \Pi^* \bar{v}), w) = 0 \quad \text{for all } w \in W^N.$$

Consequently, the choice $\Pi = \Pi^*$ would eliminate one more term in the error equation and thus in (4.7). But do we have for the projection error the desired estimate

$$(4.9) \quad \varepsilon^{1/4} \|\Pi^*(\nabla u) - \nabla u\|_0 \preceq N^{-1} \ln N \quad ?$$

The answer is no (see Lin and Stynes [13], page 2738). The reason lies in the fact that Π^* is applied to ∇u and its behavior near the transition point of the mesh is different from the behavior of u (a factor ε^{-1}).

Therefore, Lin and Stynes define a modified interpolant $\Pi \bar{v} \in W^N$, such that

$$(4.10) \quad \varepsilon^{1/4} \|\Pi(\nabla u) - \nabla u\|_0 \preceq N^{-1} \ln N$$

([13], Corollary 4.6). The operator Π is defined differently for every component of the solution decomposition. For the smooth part one takes simply $\Pi = \Pi^*$.

For the layer components, however, Π^* is modified. Consider, for instance, the layer component w_1 related to $\exp(-\sqrt{c}y/\sqrt{\varepsilon})$. Then Π and Π^* differ only in the small strip R_1 defined by

$$R_1 := [0, 1] \times [\lambda - h^*, \lambda] \quad \text{with} \quad \lambda = 2\sqrt{\varepsilon} \ln N / \sqrt{c} \quad \text{and} \quad h^* = O(\sqrt{\varepsilon} N^{-1} \ln N).$$

On that strip we loose the property (4.8), therefore we have additionally to estimate

$$(4.11) \quad M_{1,R_1} := \varepsilon^{1/2} \|\nabla \cdot (\Pi(\nabla w_1) - \nabla w_1)\|_{0,R_1}.$$

On R_1 we have $\|\Delta w_1\| \preceq \varepsilon^{-1} N^{-2}$, consequently

$$(4.12) \quad \varepsilon^{1/2} \|\Delta w_1\|_{0,R_1} \preceq \varepsilon^{-1} N^{-2} \varepsilon^{1/4} N^{-1/2} (\ln N)^{1/2} = \varepsilon^{-1/4} N^{-5/2} (\ln N)^{1/2}.$$

By construction the components of $\Pi(\nabla w_1)$ satisfy $(\Pi(\nabla w_1))_1 = 0$ on R_1 and $\|(\Pi(\nabla w_1))_2\|_\infty \preceq \varepsilon^{-1} N^{-2}$. It follows

$$(4.13) \quad \varepsilon^{1/2} \|\nabla \cdot (\Pi \nabla w_1)\|_{0,R_1} \preceq \varepsilon^{1/2} \frac{1}{h^*} \varepsilon^{-1/2} N^{-2} (h^*)^{1/2} = \varepsilon^{-1/4} N^{-3/2} (\ln N)^{-1/2}.$$

Therefore

$$(4.14) \quad M_{1,R_1} \preceq \varepsilon^{-1/4} N^{-3/2}.$$

The other layer components of the solution decomposition of u are treated similarly. We obtain finally

$$(4.15) \quad \varepsilon^{1/4} \|\Pi \bar{q} - \bar{q}^N\|_0 \preceq N^{-1} \ln N$$

and

$$(4.16) \quad \varepsilon^{1/4} \|\nabla u - \bar{q}^N\|_0 \preceq N^{-1} \ln N.$$

Remark 1. *It is well known [3], [2] that mixed methods can be reformulated as non-mixed formulations, more precisely as projected nonconforming methods. This allows as well error estimates for certain nonconforming methods as the implementation of a mixed method as nonconforming method.*

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